

Position and Orientation Based Formation Control of Multiple Rigid Bodies with Collision Avoidance and Connectivity Maintenance

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Abstract—This paper addresses the problem of position- and orientation-based formation control of a class of second-order nonlinear multi-agent systems in a 3D workspace with obstacles. More specifically, we design a decentralized control protocol such that each agent achieves a predefined geometric formation with its initial neighbors, while using local information based on a limited sensing radius. The latter implies that the proposed scheme guarantees that the initially connected agents remain always connected. In addition, by introducing certain distance constraints, we guarantee inter-agent collision avoidance as well as collision avoidance with the obstacles and the boundary of the workspace. The proposed controllers employ a novel class of potential functions and do not require a priori knowledge of the dynamical model, except for gravity-related terms. Finally, simulation results verify the validity of the proposed framework.

I. INTRODUCTION

During the last decades, decentralized control of multi-agent systems has gained a significant amount of attention due to the great variety of its applications, including multi-robot systems, transportation, multi-point surveillance and biological systems. The main focus of multi-agent systems is the design of distributed control protocols in order to achieve global tasks, such as consensus, and at the same time fulfill certain properties, e.g., network connectivity.

A particular multi-agent problem that has been considered in the literature is the formation control problem, where the agents represent robots that aim to form a prescribed geometrical shape, specified by a certain set of desired relative configurations. The main categories of formation control that have been studied in the related literature are ([1]) position-based control, displacement-based control, distance-based control and orientation-based control.

In position-based formation control, the agents control their positions to achieve the desired formation, as prescribed by some desired position offsets with respect to a global coordinate system. When orientation alignment is considered as a control design goal, the problem is known as orientation-based (or bearing-based) formation control. The desired formation is then defined by relative inter-agent orientations. The orientation-based control steers the agents to configurations that achieve desired relative orientation angles. In this work, we aim to design decentralized control protocols

such that both position- and orientation-based formation are achieved.

The literature in position-based formation control is rich, and is traditionally categorized in single or double integrator agent dynamics and directed or undirected communication topologies (see e.g. [2]–[15]). Orientation-based formation control has been addressed in [16]–[19], whereas the authors in [19]–[21] have considered the combination of position- and orientation-based formation.

The dominant case in the related literature of formation control is the two-dimensional one with simple dynamics and point-mass agents. In real applications, however, the engineering systems may have nonlinear 2nd order dynamics, for which due to imperfect modeling the exact model is not a priori known. Other objectives concern connectivity maintenance, collision avoidance between the agents as well as collision avoidance between the agents and potential obstacles of the workspace, which renders the formation control problem a particularly challenging task. According to the authors' best knowledge, the combination of the aforementioned specifications has not been addressed in the related literature.

Motivated by this, we aim to address here the position-based formation control problem with orientation alignment for a team of rigid bodies operating in 3D space, with 2nd order nonlinear dynamics. We propose a decentralized control protocol that guarantees a geometric prescribed position- and orientation-based formation between initially connected agents. The proposed methodology guarantees inter-agent collision avoidance and collision avoidance with the obstacles and the boundary of the workspace. In parallel, connectivity maintenance of the initially connected agents as well as representation singularity avoidance are ensured. In order to deal with the aforementioned specifications, we employ a novel class of potential functions. A special case of correct-by-construction potential functions, namely *navigation functions*, has been introduced in [22] for the single-robot navigation, and has been employed in multi-agent formation control in [23]–[28]. A more general potential function framework has been employed in [5]. The aforementioned works, however, have only addressed the single integrator case, with no straightforward extension to higher-order systems. The authors in [29] deal with the double integrator case, but the goal was only navigation of the agents to specific points. In addition, in many works that employ navigation functions for navigation/formation control, the designed gains and parameters (usually referred as k -the navigation function gain- and ε -the arbitrarily small distance to the obstacles) cannot be found trivially, since,

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they appear in both sides of the derived inequalities.

In our previous work [30], we treated a similar problem by utilizing a Prescribed Performance Control (PPC) scheme instead (for PPC controller design we refer to [31]), while only guaranteeing collision avoidance between neighboring agents forming a tree, with no obstacles or representation singularity avoidance. The main contribution of this paper is a novel decentralized control protocol scheme that generalizes [30] and solves a wider class of problems of multiple rigid bodies under Lagrangian dynamics with guaranteed collision avoidance among the agents, collision avoidance between agents and obstacles as well as singularity avoidance.

The remainder of the paper is structured as follows. Section II gives the necessary notation. Section III provides the system dynamics and the formal problem statement. Section IV discusses the technical details of the solution and Section V is devoted to a simulation example. Finally, conclusions and future work are discussed in Section VI.

II. NOTATION

The set of positive integers is denoted by \mathbb{N} . The real n -coordinate space, with $n \in \mathbb{N}$, is denoted by \mathbb{R}^n ; $\mathbb{R}_{\geq 0}^n$ and $\mathbb{R}_{>0}^n$ are the sets of real n -vectors with all elements nonnegative and positive, respectively. Given a set S , we denote by $|S|$ its cardinality and by $S^N = S \times \dots \times S$ its N -fold Cartesian product. The notation $\|x\|$ is used for the Euclidean norm of a vector $x \in \mathbb{R}^n$. Define by $I_n \in \mathbb{R}^{n \times n}$, $0_{m \times n} \in \mathbb{R}^{m \times n}$ the identity matrix and the $m \times n$ matrix with all entries zeros, respectively. A matrix $S \in \mathbb{R}^{n \times n}$ is called skew-symmetric if and only if $S^\top = -S$; $\mathcal{B}(c, r) = \{x \in \mathbb{R}^3 : \|x - c\| \leq r\}$ is the 3D sphere of center $c \in \mathbb{R}^3$ and radius $r \in \mathbb{R}_{>0}$. Given a scalar function $y : \mathbb{R}^n \rightarrow \mathbb{R}$ and a vector $x \in \mathbb{R}^n$, denote by $\nabla_x y(x) = \frac{\partial y(x)}{\partial x} = [\frac{\partial y(x)}{\partial x_1}, \dots, \frac{\partial y(x)}{\partial x_n}]^\top \in \mathbb{R}^n$ the gradient of y . The vector connecting the origins of coordinate frames $\{A\}$ and $\{B\}$ expressed in frame $\{C\}$ coordinates in 3D space is denoted by $p_{B/A}^C \in \mathbb{R}^3$. We further denote by $q_{B/A} = [\phi_{B/A}, \theta_{B/A}, \psi_{B/A}]^\top \in \mathbb{T}^3$ the Euler angles representing the orientation of frame $\{B\}$ with respect to frame $\{A\}$, with $-\pi \leq \phi_{A/B}, \psi_{A/B} \leq \pi$ and $\frac{\pi}{2} \leq \theta_{A/B} \leq \frac{\pi}{2}$, where \mathbb{T}^3 is the 3D torus. The angular velocity of frame $\{B\}$ with respect to $\{A\}$, expressed in frame $\{C\}$ coordinates, is denoted by $\omega_{B/A}^C \in \mathbb{R}^3$. We also use the notation $\mathbb{M} = \mathbb{R}^3 \times \mathbb{T}^3$. For notational brevity, when a coordinate frame corresponds to an inertial frame of reference $\{0\}$, we will omit its explicit notation (e.g., $p_B = p_{B/0}^0, \omega_B = \omega_{B/0}^0$). All vector and matrix differentiations are derived with respect to the inertial frame $\{0\}$, unless otherwise stated.

III. PROBLEM FORMULATION

A. System Model

Consider a set of N rigid bodies, with $\mathcal{V} = \{1, 2, \dots, N\}$, $N \geq 2$, operating in a workspace $W \subseteq \mathbb{M}$, with coordinate frames $\{i\}, i \in \mathcal{V}$, attached to their centers of mass. The workspace is assumed to be modeled as a bounded sphere $W = \mathcal{B}(p_w, r_w)$ with center p_w and radius r_w . Without loss of generality, we assume that $p_w = 0_{3 \times 1}$, representing an inertial reference frame $\{0\}$. The subscript w stands for the

workspace W . We consider that each agent occupies a sphere $\mathcal{B}(p_i(t), r_i)$, where $p_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^3$ is the position of the agent's center of mass and $r_i < r_w$ is the agent's radius. We also denote by $q_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{T}^3, i \in \mathcal{V}$, the Euler angles representing the agents' orientation with respect to $\{0\}$, with $q_i = [\phi_i, \theta_i, \psi_i]^\top$. By defining $x_i : \mathbb{R}_{\geq 0} \rightarrow W, v_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^6$, with $x_i = [p_i^\top, q_i^\top]^\top, v_i = [\dot{p}_i^\top, \dot{\omega}_i^\top]^\top$, we model each agent's motion with the 2nd order dynamics:

$$\dot{x}_i = J_i^{-1}(q_i)v_i, \quad (1a)$$

$$u_i = M_i(x_i)\dot{v}_i + C_i(x_i, \dot{x}_i)v_i + g_i(x_i), \quad (1b)$$

where $J_i : \mathbb{T}^3 \rightarrow \mathbb{R}^{6 \times 6}$ is a *Jacobian matrix* that maps the Euler angle rates to v_i , given by

$$J_i(q_i) = \begin{bmatrix} I_3 & 0_{3 \times 3} \\ 0_{3 \times 3} & J_{q_i}(q_i) \end{bmatrix},$$

$$J_{q_i}(q_i) = \begin{bmatrix} 1 & 0 & \sin(\theta_i) \\ 0 & \cos(\phi_i) & -\cos(\theta_i)\sin(\phi_i) \\ 0 & \sin(\phi_i) & \cos(\phi_i)\cos(\theta_i) \end{bmatrix},$$

and $J_i^{-1}(q_i)$ is its matrix inverse. The matrix J_i is singular when $\det(J_i) = \cos(\theta_i) = 0 \Leftrightarrow \theta_i = \pm \frac{\pi}{2}$, which we refer to as *representation singularity*. The proposed controller will guarantee, however, that this is always avoided and thus (1a) is well defined.

Furthermore, $M_i : W \rightarrow \mathbb{R}^{6 \times 6}$ is the positive definite *inertia matrix*, $C_i : W \times \mathbb{R}^6 \rightarrow \mathbb{R}^{6 \times 6}$ is the *Coriolis matrix*, and $g_i : W \rightarrow \mathbb{R}^6$ is the *gravity vector*. We consider that the Coriolis and the inertia vector fields are *unknown*. Finally, $u_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^6$ is the control input vector representing the 6D generalized *actuation force* acting on agent $i \in \mathcal{V}$. Let us also define the stack vectors $x = [x_1^\top, \dots, x_N^\top]^\top : \mathbb{R}_{\geq 0} \rightarrow W^N$ and $v = [v_1^\top, \dots, v_N^\top]^\top : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{6N}$. Moreover, the notation $\bar{x}_i(t)$ is used for the vector of neighbors of agent i at time t , i.e., $\bar{x}_i(t) = [x_{i_1}^\top(t), \dots, x_{i_{N_i}}^\top(t)]^\top$, with i_1, \dots, i_{N_i} being the neighbors of agent i at time t . Similarly, $\bar{v}_i(t) = [v_{i_1}^\top(t), \dots, v_{i_{N_i}}^\top(t)]^\top$ denotes the velocities of the neighbors of agent i . In addition, the matrices $\dot{M}_i - 2C_i$ are skew-symmetric [32], i.e., $y^\top [\dot{M}_i - 2C_i] y = 0, \forall y \in \mathbb{R}^6, i \in \mathcal{V}$. For the state measurement of each agent, the following assumption is required.

Assumption 1. (Measurements Assumption) Each agent i can measure its own states $p_i, q_i, \dot{p}_i, v_i, i \in \mathcal{V}$, and has a limited sensing range of: $d_i > \max\{r_i + r_j : i, j \in \mathcal{V}, i \neq j\}$.

Therefore, by defining the neighboring set as $\mathcal{N}_i(t) = \{j \in \mathcal{V} : p_j(t) \in \mathcal{B}(p_i(t), d_i), i \neq j\}$, in view of the aforementioned assumption, agent i knows at each time instant t all $p_{j/i}^i(t), q_{j/i}^i(t), \dot{p}_{j/i}^i(t), \omega_{j/i}^i(t)$ and, since it knows its own $p_i(t), q_i(t)$, it can compute all $p_j(t), q_j(t), \dot{p}_j(t), \omega_j(t), \forall j \in \mathcal{N}_i(t), t \in \mathbb{R}_{\geq 0}$. For the neighboring set $\mathcal{N}_i(t)$ define also $N_i(t) = |\mathcal{N}_i(t)|$.

Moreover, we consider that in the given workspace there exist $Z \in \mathbb{N}$ *static obstacles*, with $\mathcal{Z} \triangleq \{1, \dots, Z\}$, modeled as the spheres $\mathcal{B}(p_{o_z}, r_{o_z})$, with centers and radii $p_{o_z} \in \mathbb{R}^3, r_{o_z} \in \mathbb{R}_{>0}, z \in \mathcal{Z}$, respectively. The geometry of two agents i, j and an obstacle z in the workspace W is depicted in Fig. 1.

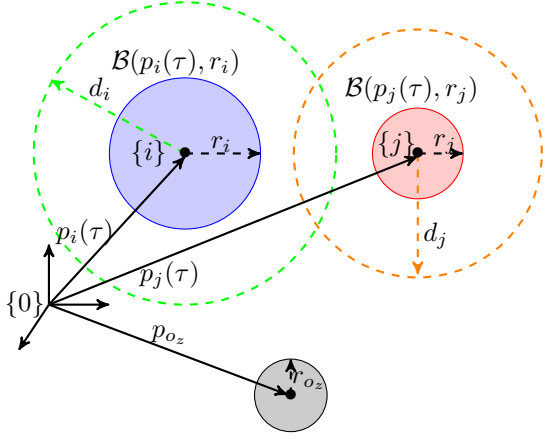


Fig. 1: Illustration of two moving agents $i, j \in \mathcal{V}$ and a static obstacle o_z in the workspace for a time instant τ ; $\{0\}$ is the inertial frame, $\{i\}, \{j\}$ are the frames attached to the agents' center of mass, $p_i(\tau), p_j(\tau), p_{o_z} \in \mathbb{R}^3$ are the positions of the center of mass of the agents i, j and the obstacle o_z , respectively, with respect to $\{0\}$; r_i, r_j, r_{o_z} are the radii of the agents i, j and the obstacle o_z , respectively; d_i, d_j with $d_i > d_j$ are the agents' sensing ranges. Note that the agents are not neighbors since $p_j(t) \notin \mathcal{B}(p_i(\tau), d_i)$ and $p_i(\tau) \notin \mathcal{B}(p_j(\tau), d_j)$.

Let us define the distances $d_{ij,a} : \mathbb{R}^6 \rightarrow \mathbb{R}_{\geq 0}$, $d_{iz,o} : \mathbb{R}^3 \rightarrow \mathbb{R}_{\geq 0}$, with: $d_{ij,a}(p_i, p_j) = \|p_i - p_j\|$, $d_{iz,o}(p_i) = \|p_i - p_{o_z}\|$, $\forall i, j \in \mathcal{V}, i \neq j, z \in \mathcal{Z}$, as well as the constants $\underline{d}_{ij,a} = r_i + r_j$, $\underline{d}_{iz,o} = r_i + r_{o_z}$, that represent the minimum distance such that agents i and j and agent i and object z , do not collide, respectively. The subscripts a and o stand for *agent* and *obstacle*, respectively. The following assumption is required, for the feasibility of the problem:

Assumption 2. It holds that

- 1) $\|p_{o_z} - p_{o_z'}\| \geq 2 \max_{i \in \mathcal{V}} \{r_i\} + r_{o_z} + r_{o_z'} + \varepsilon_r, \forall z, z' \in \mathcal{Z}$, with $z \neq z'$,
- 2) $r_w - (\|p_{o_z}\| + r_{o_z}) \geq 2 \max_{i \in \mathcal{V}} \{r_i\} + \varepsilon_r, \forall z \in \mathcal{Z}$,

where ε_r is an arbitrarily small positive constant.

The aforementioned assumption states that there is enough space between the obstacles and the workspace boundary as well as the obstacles themselves for the agents to navigate among them.

B. Problem Statement

Due to the fact that the agents are not dimensionless and their sensing capabilities are limited, the control protocol, except from achieving desired position formation (define it by $p_{ij,\text{des}}$) and desired formation angles (define it by $q_{ij,\text{des}}$) for all neighboring agents $i \in \mathcal{V}, j \in \mathcal{N}_i(0)$, it should also guarantee for all $t \in \mathbb{R}_{\geq 0}$ that (i) all the agents avoid collision with every other agent; (ii) all the agents avoid collision with all the obstacles; (iii) all the agents avoid collision with the workspace boundary, (iv) all the initial edges are maintained, i.e., connectivity maintenance, and (v) the singularity of the Jacobian matrices J_i is avoided.

Definition 1. (Feasible Formation) Given the initial neighboring sets $\mathcal{N}_i(0), i \in \mathcal{V}$, the desired displacements $x_{ij,\text{des}} = [p_{ij,\text{des}}^\top, q_{ij,\text{des}}^\top]^\top$ that characterize a formation configuration,

are called *feasible* if $\cap_{i \in \mathcal{V}} \{x_i \in W : \|x_i - x_j - x_{ij,\text{des}}\| = 0, d_{iz,o}(p_i) > 0, \|p_i\| + r_i < r_w, \forall z \in \mathcal{Z}, j \in \mathcal{N}_i(0)\} \neq \emptyset$.

Formally, the control problem under the aforementioned constraints is formulated as follows:

Problem 1. Consider N agents governed by the dynamics (1) and operating in a workspace W with Z spherical obstacles, with:

- $v_i(0) = 0_{6 \times 1}, \forall i \in \mathcal{V}$,
- $-\frac{\pi}{2} < -\bar{\theta} \leq \theta_i(0) \leq \bar{\theta} < \frac{\pi}{2}, \forall i \in \mathcal{V}$,
- $\|p_i(0) - p_j(0)\| > \underline{d}_{ij,a}, \forall i, j \in \mathcal{V}, i \neq j$,
- $\|p_i(0) - p_{o_z}(0)\| > \underline{d}_{iz,o}, \forall i \in \mathcal{V}, z \in \mathcal{Z}$,

i.e., singularity- and collision- free configurations at $t = 0$, where $\bar{\theta}$ is an arbitrary constant in the open set $(0, \frac{\pi}{2})$. Then, given a nonempty initial set $\mathcal{N}_i(0) \neq \emptyset$, *feasible* inter-agent displacements $p_{ij,\text{des}}, q_{ij,\text{des}}, \forall i \in \mathcal{V}, j \in \mathcal{N}_i(0)$, such that $\underline{d}_{ij,a} < d_{ij,a}(p_i, p_j) < d_i, \forall (p_i, p_j) \in \{(p_i, p_j) \in \mathbb{R}^6 : \|x_i - x_j - x_{ij,\text{des}}\| = 0\}$, design decentralized control laws $u_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^6$, such that for every $i \in \mathcal{V}$ the following hold:

- 1) $\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t) - x_{ij,\text{des}}\| < \mu_i, \forall j \in \mathcal{N}_i(0)$,
- 2) $\|p_i(t) - p_j(t)\| > \underline{d}_{ij,a}, \forall j \in \mathcal{V} \setminus \{i\}, t \in \mathbb{R}_{\geq 0}$,
- 3) $\|p_i(t) - p_{o_z}(t)\| > \underline{d}_{iz,o}, \forall z \in \mathcal{Z}, t \in \mathbb{R}_{\geq 0}$,
- 4) $\|p_i(t)\| + r_i < r_w, \forall t \in \mathbb{R}_{\geq 0}$.
- 5) $\|p_i(t) - p_j(t)\| < d_i, \forall j \in \mathcal{N}_i(0), t \in \mathbb{R}_{\geq 0}$,
- 6) $-\frac{\pi}{2} < -\bar{\theta} \leq \theta_i(t) \leq \bar{\theta} < \frac{\pi}{2}, \forall t \in \mathbb{R}_{\geq 0}$,

where μ_i is an arbitrarily small $\forall i \in \mathcal{V}$.

The aforementioned specifications imply the following: 1) stands for formation control (both position and orientation); 2) stands for inter-agent collision avoidance; 3) stands for collision avoidance between the agents and the obstacles; 4) stands for collision avoidance between the agents and the boundary; 5) stands for connectivity maintenance of the initially connected agents and finally, 6) stands for the representation of singularities avoidance.

IV. PROBLEM SOLUTION

In this section, a systematic solution to Problem 1 is introduced. Our overall approach builds on designing a decentralized potential function for each agent that captures all the desired control specifications. This potential function will then be exploited by the decentralized controller of each agent. In particular, the following analysis is performed: 1) The form of the proposed potential function along with its components is described in Section IV-A. 2) The proposed decentralized controllers that guarantee the satisfaction of all the control specifications are provided in Section IV-B. The required stability analysis is presented subsequently.

A. Decentralized Potential Functions

In order to solve the formation control problem with the collision- and singularity- avoidance as well as connectivity

maintenance, we define a *decentralized potential function* for each agent $i \in \mathcal{V}$ as:

$$\varphi_i(x) = k_i \gamma_i(x) + \frac{1}{\beta_i(x)}, \quad (2)$$

where:

- $\gamma_i(x) : W^N \rightarrow \mathbb{R}_{\geq 0}$ is the *goal function* that vanishes when agent i is at the desired position and orientation with its neighbors i_1, \dots, i_{N_i} .
- $\beta_i(x) : W^N \rightarrow \mathbb{R}$, is an *obstacle function* that encodes collisions between agents and obstacles, collisions between agents and the obstacle boundary, connectivity losses between initially connected agents and singularities of the Jacobian matrix $J_i(x_i)$; $\beta_i(x)$ vanishes when one or more of the above situations occurs.
- $k_i \in \mathbb{N}$ is a tuning parameter that will be explicitly defined later for guaranteeing the stability properties of the multi-agent system.

The general form of the proposed potential function φ_i is motivated by the initial work of the correct-by-construction navigation functions [22]. The function φ_i proposed in this work, however, is not argued to be a navigation function. Moreover, due to Assumption 2, the domain of φ_i is connected. In the sequel, we describe the construction of a function of the form (2).

1) $\gamma_i(x)$ - *Goal Function*: The function $\gamma_i : W^N \rightarrow \mathbb{R}_{\geq 0}$ encodes the control objective of agent i , which is to achieve position and orientation formation with its neighboring agents. With that in mind, a reasonable choice of $\gamma_i(x)$ is:

$$\begin{aligned} \gamma_i(x) &= \sum_{j \in \mathcal{N}_i(0)} \{\gamma_{ij,p}(p_i, p_j) + \gamma_{ij,q}(q_i, q_j)\}, \\ &= \sum_{j \in \mathcal{N}_i(0)} \gamma_{ij,x}(x_i, x_j), \end{aligned} \quad (3)$$

where $\gamma_{ij,p}(p_i, p_j) = \rho_{ij} \|p_i - p_j - p_{ij,\text{des}}\|^2$, $\gamma_{ij,q}(q_i, q_j) = \rho_{ij} \|q_i - q_j - q_{ij,\text{des}}\|^2$, $\gamma_{ij,x}(x_i, x_j) = \rho_{ij} \|x_i - x_j - x_{ij,\text{des}}\|^2$ and ρ_{ij} are positive gains that will be defined later. The function $\gamma_i(x)$ reaches its unique global minimum when both $p_i - p_j = d_{ij,\text{des}}$ and $q_i - q_j = q_{ij,\text{des}}$, $\forall i \in \mathcal{V}$, i.e., when both formation and orientation alignment are achieved between all the neighboring agents.

2) $\beta_i(x)$ - *Obstacle Function*: The function $\beta_i(x) : W^N \rightarrow \mathbb{R}$, encodes all inter-agent collisions, collisions between the agents and obstacles, collisions with the boundary of the workspace, connectivity between initially connected agents and singularities of the Jacobian matrix $J_i(x_i)$. First, for each agent i , we define the functions $\eta_{ij,a} : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$, $\eta_{iz,o} : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$, $\eta_{ij,c} : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ where:

$$\eta_{ij,a}(d_{ij,a}) = d_{ij,a}^2 - \underline{d}_{ij,a}^2, \quad (4a)$$

$$\eta_{iz,o}(d_{iz,o}) = d_{iz,o}^2 - \underline{d}_{iz,o}^2, \quad (4b)$$

$$\eta_{ij,c}(d_{ij,a}) = d_i^2 - \underline{d}_{ij,a}^2. \quad (4c)$$

The subscripts j and z correspond to agent $j \in \mathcal{V} \setminus \{i\}$ and obstacle $z \in \mathcal{Z}$, respectively, whereas the subscript c stands for *connectivity*. Let us also define the functions $b_{ij,a} :$

$\mathbb{R}_{\geq 0} \rightarrow [0, 1]$, $b_{iz,o} : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$, $b_{ij,c} : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$, $b_{iw} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, $b_{J_i} : [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow [0, 1]$, where:

$$b_{ij,a}(x) = \begin{cases} \phi_{i,a}(x), & 0 \leq x < d_i^2 - \underline{d}_{ij,a}^2, \\ 1, & d_i^2 - \underline{d}_{ij,a}^2 \leq x, \end{cases} \quad (5a)$$

$$b_{iz,o}(x) = \begin{cases} \phi_{i,o}(x), & 0 \leq x < d_i^2 - \underline{d}_{iz,o}^2, \\ 1, & d_i^2 - \underline{d}_{iz,o}^2 \leq x, \end{cases} \quad (5b)$$

$$b_{ij,c}(x) = \begin{cases} 0 & x < 0, \\ \phi_{i,c}(x), & 0 \leq x < d_i^2 - \underline{d}_{ij,a}^2, \\ 1, & d_i^2 - \underline{d}_{ij,a}^2 \leq x, \end{cases} \quad (5c)$$

$$b_{iw}(x) = \left[1 - \frac{x}{(r_w - r_i)^2} \right]^2, \quad (5d)$$

$$b_{J_i}(x) = \cos^2(x). \quad (5e)$$

The functions $\phi_{i,a}, \phi_{i,o}, \phi_{i,c}$ are *increasing polynomials*, appropriately selected to guarantee that the functions $b_{ij,a}, b_{iz,o}, b_{ij,c}$, respectively, are twice continuously differentiable everywhere, with $\phi_{i,a}(0) = \phi_{i,o}(0) = \phi_{i,c}(0) = 0$, $\forall i \in \mathcal{V}$. An example of a function $b_{ij,a}$ along with its derivatives $\frac{\partial b_{ij,a}(x)}{\partial x}$, $\frac{\partial^2 b_{ij,a}(x)}{\partial x^2}$ is depicted in Fig. 2. The functions $b_{iz,o}, b_{ij,c}$ have an identical behavior. We can now choose the function $\beta_i : W^N \rightarrow [0, 1]$ as the following product for every $i \in \mathcal{V}$:

$$\beta_i(x) = b_{iw}(\|p_i\|^2) b_{J_i}(\theta_i) \left[\prod_{j \in \mathcal{V} \setminus \{i\}} b_{ij,a}(\eta_{ij,a}) \right] \left[\prod_{z \in \mathcal{Z}} b_{iz,o}(\eta_{iz,o}) \right] \left[\prod_{j \in \mathcal{N}_i(0)} b_{ij,c}(\eta_{ij,c}) \right]. \quad (6)$$

The functions $b_{ij,a}(\eta_{ij,a}), b_{iz,o}(\eta_{iz,o}), b_{ij,c}(\eta_{ij,c})$ correspond to inter-agent collision, collision with obstacles and connectivity maintenance, respectively, for agent $i \in \mathcal{V}$, while the functions $b_{iw}(\|p_i\|^2), b_{J_i}(\theta_i)$ correspond to collision with the workspace boundary and representation singularities. Each of these terms becomes zero when there is a collision, a connectivity break or a representation singularity. Note that all the aforementioned functions use only local information depending on the sensing range d_i of agent i .

Remark 1. Note that the choice of the functions γ_i and β_i from (3) and (6), respectively, renders φ_i from (3), (6) to be a function only of the neighboring states x_i and \bar{x}_i , i.e., decentralized. The function γ_i depends on the initial neighboring set $\mathcal{N}_i(0)$ since the formation requirements need to be achieved between the agents that belong to the initial neighboring set $\mathcal{N}_i(0)$. Furthermore, the function β_i depends on the time-varying neighboring set $\mathcal{N}_i(t)$ for every $t \in \mathbb{R}_{\geq 0}$, since in order to capture the collision avoidance goals, the neighboring sets $\mathcal{N}_i(t)$ need to be updated with new potential neighbors.

Remark 2. As will be shown later, by guaranteeing all the desired specifications, the obstacle functions $\beta_i(x)$ remain strictly positive and upper bounded by 1, i.e., $0 < \beta_i(x(t)) \leq 1, \forall x \in W^N, t \in \mathbb{R}_{\geq 0}$.

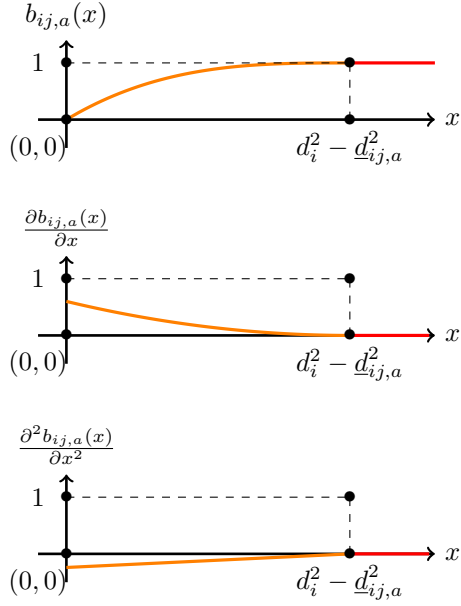


Fig. 2: The functions $b_{ij,a}(x)$ (top), $\frac{\partial b_{ij,a}(x)}{\partial x}$ (middle) and $\frac{\partial^2 b_{ij,a}(x)}{\partial x^2}$ (bottom), for $d_i^2 - d_{ij,a}^2 = 5$ and $\phi_{i,a}(x) = 0.008x^3 - 0.12x^2 + 0.6x$.

B. Control Design and Stability Analysis

Let us define the following sets:

$$\begin{aligned}\hat{\Xi}_i &= \{x \in W^N : \gamma_{ij,x}(x) > \mu_{ij}, \forall j \in \mathcal{N}_i(0)\}, \\ \Xi_{f,i} &= \{x \in W^N : \gamma_{ij,x}(x) \leq \mu_{ij}, \forall j \in \mathcal{N}_i(0)\} = W^N \setminus \hat{\Xi}_i,\end{aligned}$$

with μ_{ij} arbitrarily small positive constants, with $\sum_{j \in \mathcal{N}_i(0)} \mu_{ij} = \mu_i, \forall i \in \mathcal{V}$. The aforementioned sets correspond to the undesired and desired configuration sets, respectively, and will be used for proving the stability of the system. Moreover, the following two lemmas are required for the subsequent analysis.

Lemma 1. *There exist positive and finite constants $\bar{\rho}_i$ and $\tilde{\rho}_i$, such that, if $\rho_{ij} = 1, \forall j \in \mathcal{N}(0) \setminus \{j_i^*\}$, and*

$$\begin{aligned}\rho_{ij_i^*} &\geq \bar{\rho}_i \triangleq \\ \tilde{\rho}_i &+ \frac{1}{\sqrt{\mu_{ij_i^*}}} \sup_{x \in \hat{\Xi}_i} \left\{ \left\| \sum_{j \in \mathcal{N}_i(0) \setminus j_i^*} (x_i - x_j - x_{ij,des}) \right\| \right\},\end{aligned}\quad (8)$$

for some initial neighboring agent j_i^* in $\mathcal{N}_i(0)$ of agent i , then

$$\|\nabla_{x_i} \gamma_i(x)\| \geq \tilde{\rho}_i > 0, \forall x \in \hat{\Xi}_i, i \in \mathcal{V}. \quad (9)$$

Proof. The proof can be found in Appendix I. \square

Remark 3. The choice of j_i^* is arbitrary among the initial neighbors of agent i . Moreover, although we set the constants $\rho_{ij}, \forall j \in \mathcal{N}(0) \setminus \{j_i^*\}$ equal to 1, they can be arbitrary finite positive constants, as can be concluded from the proof in Appendix I.

Lemma 2. *Suppose that there exists a positive constant $\underline{\beta}$ such that $\beta_i(x) \geq \underline{\beta} > 0, \forall i \in \mathcal{V}$. Then, there exists a*

positive constant $\bar{\beta}_i$ such that:

$$\sup_{x \in \hat{\Xi}_i} \{\|\nabla_{x_i} \beta_i(x)\|\} \leq \bar{\beta}_i, i \in \mathcal{V}. \quad (10)$$

Proof. The proof can be found in Appendix II. \square

Next, we design bounded controllers u_i such that all the specifications of Problem 1 are met, according to the following theorem, which summarizes the main results of this work.

Theorem 1. *Suppose that Assumptions 1, 2 hold. Then, there exist positive and finite \underline{k}_i , such that if $k_i > \underline{k}_i, \forall i \in \mathcal{V}$, the decentralized control law $u_i : W \times W^{N_i} \times \mathbb{R}^6 \times \mathbb{R}^{6N_i} \rightarrow \mathbb{R}^6$, with*

$$\begin{aligned}u_i(x_i, \bar{x}_i, v_i, \bar{v}_i) &= -\tilde{k}_i v_i + g_i(x_i) \\ &- \frac{\tilde{c}_i v_i}{\tanh(\|v_i\|^2)} \left[\sum_{j \in \mathcal{N}_i(t)} [\nabla_{x_j} \varphi_i(x)]^\top J_j^{-1}(x_j) v_j \right. \\ &\quad \left. - [J_i^{-1}(x_i)]^\top \nabla_{x_i} \varphi_i(x), \right]\end{aligned}\quad (11)$$

for each agent $i \in \mathcal{V}$, with control gains $\tilde{c}_i > 1, \tilde{k}_i > 0$, brings agent i to the desired set $\Xi_{f,i}$ while ensuring $\beta_i > 0, \forall i \in \mathcal{V}$ and the boundedness of all closed loop signals, providing, thus, a solution to Problem 1.

Proof. Let us first define the vectors $\gamma = [\gamma_1, \dots, \gamma_N]^\top$ and $\beta = [\frac{1}{\beta_1}, \dots, \frac{1}{\beta_N}]^\top$. Consider the following nonnegative Lyapunov-like function for the system (1):

$$L(\gamma, \beta, v, x) = \sum_{i \in \mathcal{V}} \left\{ k_i \gamma_i + \frac{1}{\beta_i} + \frac{1}{2} v_i^\top M_i(x_i) v_i \right\}, \quad (12)$$

which is well defined, since the domain of $\frac{1}{\beta_i}$ is connected, $\forall i \in \mathcal{V}$, according to Assumption 2. Since the system configuration at $t = 0$ is singularity- and collision-free, the functions β_i are strictly positive at $t = 0, \forall i \in \mathcal{V}$. Furthermore, $v_i(0) = 0_{6 \times 1}, \forall i \in \mathcal{V}$. Thus, L is initially bounded, i.e., there exists a positive and finite constant $M > 1$ such that

$$L_0 \triangleq L(\gamma(x(0)), \beta(x(0)), v(0), x(0)) \leq M. \quad (13)$$

By differentiating (12) with respect to time, substituting the dynamics (1), using the skew-symmetry of $\dot{M}_i - 2C_i, \forall i \in \mathcal{V}$ and grouping terms, we obtain:

$$\begin{aligned}\dot{L} &= \sum_{i \in \mathcal{V}} \left\{ v_i^\top \left[(J_i^{-1}(x_i))^\top \nabla_{x_i} \varphi_i(x) + u_i - g_i(x_i) \right] \right. \\ &\quad \left. + \sum_{j \in \mathcal{N}_i(t)} [\nabla_{x_j} \varphi_i(x)]^\top \dot{x}_j \right\},\end{aligned}$$

which, by substituting the control law (11) and invoking the fact that $y \leq |y|, \forall y \in \mathbb{R}$, becomes:

$$\begin{aligned}\dot{L} &\leq - \sum_{i \in \mathcal{V}} \tilde{k}_i \|v_i\|^2 \\ &- \sum_{i \in \mathcal{V}} \left[\frac{\tilde{c}_i \|v_i\|^2}{\tanh(\|v_i\|^2)} - 1 \right] \left| \sum_{j \in \mathcal{N}_i(t)} [\nabla_{x_j} \varphi_i(x)]^\top \dot{x}_j \right|.\end{aligned}\quad (14)$$

Moreover, by employing the fact that $\tilde{c}_i \frac{x^2}{\tanh(x^2)} \geq \tilde{c}_i > 1, \forall x \in \mathbb{R}, \tilde{c}_i > 1$, we obtain from (14) that $\dot{L} \leq -\sum_{i \in \mathcal{V}} k_i \|v_i\|^2 \leq 0$. Therefore, L is non-increasing and hence, in view of (13), we conclude that

$$L(\gamma(x(t)), \beta(x(t)), v(t), x(t)) \leq L_0 \leq M, \forall t \in \mathbb{R}_{\geq 0}. \quad (15)$$

and the boundedness of all the terms $x(t), \beta(x(t)), \gamma(x(t)), v(t), \forall t \in \mathbb{R}_{\geq 0}$. From (15) and (12), we obtain:

$$\frac{1}{\beta_i(x)} \leq M \Leftrightarrow \beta_i(x) \geq \underline{\beta} \triangleq \frac{1}{M} > 0, \quad (16)$$

$\forall t \in \mathbb{R}_{\geq 0}, i \in \mathcal{V}$, since $M > 1$ and $k_i > 1, \forall i \in \mathcal{V}$. Hence, inter-agent collisions, collisions between the agents and the obstacles/workspace boundary as well as connectivity losses and singularities, are avoided. In addition, it follows that $\beta_{iw}(\|p_i\|^2)$ remains always strictly positive, which, in view of (5d), implies that $\|p_i\|^2 \leq (r_w - r_i)^2$ and hence $0 < b_{iw}(\|p_i(t)\|^2) \leq 1, \forall t \in \mathbb{R}_{\geq 0}$. Thus, we also conclude that $\beta_i(x(t)) \leq 1, \forall t \in \mathbb{R}_{\geq 0}, i \in \mathcal{V}$.

Moreover, by invoking LaSalle's Invariance Principle, the state of the system converges to the largest invariant set contained in the set:

$$S = \left\{ x, v : \dot{L}(x, v) = 0 \right\} = \left\{ x, v : v_i = 0_{6 \times 1}, \forall i \in \mathcal{V} \right\}. \quad (17)$$

For the subset S to be invariant we require $\dot{v}_i = 0_{6 \times 1} \Rightarrow -[J_i^{-1}(x_i)]^\top \nabla_{x_i} \varphi_i(x) = 0_{6 \times 1}$. Note that, in view of (16), $J(x_i)$ is always nonsingular. Therefore, we conclude that the closed loop system will converge to the configuration where $\nabla_{x_i} \varphi_i(x) = 0_{6 \times 1}, \forall i \in \mathcal{V}$, i.e., $k_i \nabla_{x_i} \gamma_i(x) = \frac{\nabla_{x_i} \beta_i(x)}{\beta_i^2(x)}$, which, by taking norms, becomes

$$k_i \|\nabla_{x_i} \gamma_i(x)\| = \frac{\|\nabla_{x_i} \beta_i(x)\|}{\beta_i^2(x)}, \forall i \in \mathcal{V}. \quad (18)$$

We need to guarantee that (18) does not hold in the configurations that are away from the configurations where $\gamma_i = 0$, i.e., in the set $\hat{\Xi}_i$. Therefore, we show that, by tuning the parameter k_i , (18) can hold at most in the closed set $\Xi_{f,i}$, i.e., arbitrarily close to the desired formation set. In order for (18) not to hold in the set $\hat{\Xi}_i$, i.e. when $\gamma_{ij,x} > \mu_{ij}, j \in \mathcal{N}_i(0)$, we need to design k_i such that $k_i \|\nabla_{x_i} \gamma_i(x)\| > \frac{\|\nabla_{x_i} \beta_i(x)\|}{\beta_i^2(x)}, \forall x \in \hat{\Xi}_i, i \in \mathcal{V}$, which is equivalent to designing k_i such that:

$$k_i > \sup_{x \in \hat{\Xi}_i} \left\{ \frac{\|\nabla_{x_i} \beta_i(x)\|}{\beta_i^2(x)} \frac{1}{\|\nabla_{x_i} \gamma_i(x)\|} \right\}.$$

Therefore, it is sufficient to design k_i such that:

$$k_i > \sup_{x \in \hat{\Xi}_i} \left\{ \frac{1}{\beta_i^2(x)} \right\} \sup_{x \in \hat{\Xi}_i} \left\{ \|\nabla_{x_i} \beta_i(x)\| \right\} \sup_{x \in \hat{\Xi}_i} \left\{ \frac{1}{\|\nabla_{x_i} \gamma_i(x)\|} \right\}. \quad (19)$$

By employing (16) we obtain $\sup_{x \in \hat{\Xi}_i} \left\{ \frac{1}{\beta_i^2(x)} \right\} = \frac{1}{\inf_{x \in \hat{\Xi}_i} \{\beta_i^2(x)\}} = M^2$. In addition, by employing Lemma 1, we have that $\sup_{x \in \hat{\Xi}_i} \left\{ \frac{1}{\|\nabla_{x_i} \gamma_i(x)\|} \right\} = \frac{1}{\rho_i}$. From (16), we

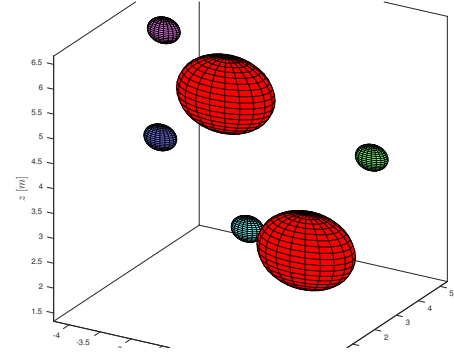


Fig. 3: The initial workspace of the simulated scenario ($t = 0$). Agent 1 (with blue), agent 2 (with green), agent 3 (with cyan) and agent 4 (with purple) and two obstacles (with red).

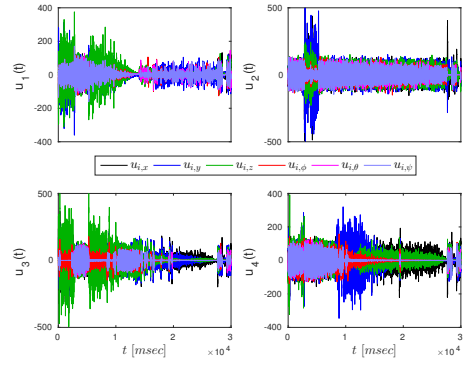


Fig. 4: The resulting control inputs $u_i, \forall i \in \{1, \dots, 4\}$.

conclude that the assumption of Lemma II holds and hence so does (10). Thus, (19) implies that

$$k_i > \underline{k}_i \triangleq \frac{\bar{\beta}_i M^2}{\rho_i}, \forall i \in \mathcal{V}. \quad (20)$$

Hence, by designing k_i as in (20), we guarantee that each agent will converge to the set $\Xi_{f,i}$, i.e., a configuration at most μ_i away from the desired configuration $\gamma_i(x) = 0, i \in \mathcal{V}$. Furthermore, the obstacle functions $\beta_i(x)$ are proved to be always lower bounded by $\frac{1}{M}$, with $M > 1$, which guarantees inter-agent collision avoidance, collision avoidance between the agents and the obstacles/workspace boundary, connectivity maintenance between the initially connected agents, and representation singularity avoidance. In addition, from (12) and (15) we conclude that the agent velocities $v_i(t)$, are bounded $\forall i \in \mathcal{V}, t \in \mathbb{R}_{\geq 0}$. Regarding the boundedness of u_i , note that we have to prove that all the velocities v_i go to $0_{6 \times 1}$ simultaneously. Assume, therefore, that there exists a nonempty set $\mathcal{V}_b \subseteq \mathcal{V}$ such that $v_i(t^*) = 0_{6 \times 1}, \forall i \in \mathcal{V}_b, v_j(t^*) \neq 0_{6 \times 1}, \forall j \in \mathcal{V} \setminus \mathcal{V}_b$, and $j \in \mathcal{N}_i(t^*), j \in \mathcal{V} \setminus \mathcal{V}_b, \forall i \in \mathcal{V}_b$, for some $t^* \in \mathbb{R}_{\geq 0}$. Then for the agents in \mathcal{V}_b , the term $\frac{\tilde{c}_i v_i(x_i)}{\tanh(\|v_i(x_i)\|^2)} | \sum_{j \in \mathcal{N}_i(t)} [\nabla_{x_j} \varphi_i(x)]^\top J_j^{-1}(x_j) v_j(x_j) |$ will result in $u_i = \infty, \forall i \in \mathcal{V}_b$ and in a non-zero velocity at the next time instant. This, however, is a contradiction, since the set S from (17) is invariant. Hence, by combining the aforementioned discussion, we can also conclude the boundedness of $u_i, \forall i \in \mathcal{V}$. Therefore, it has been proven that all the closed loop signals remain bounded $\forall i \in \mathcal{V}, t \in \mathbb{R}_{\geq 0}$, which leads to the conclusion of the proof. \square

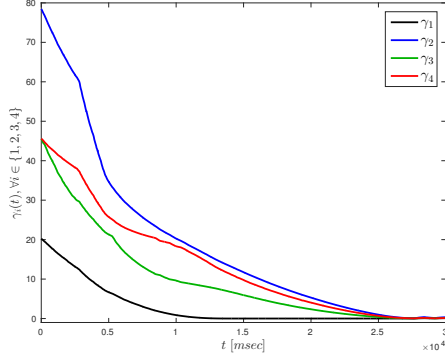


Fig. 5: The evolution of the goal functions $\gamma_i, \forall i \in \{1, \dots, 4\}$, which are shown to converge to zero.

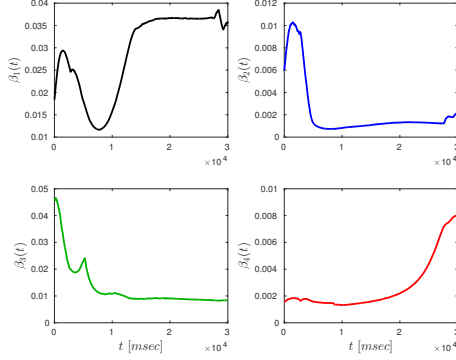


Fig. 6: The evolution of the obstacle functions $\beta_i, \forall i \in \{1, \dots, 4\}$, which are shown to be always positive.

Remark 4. Notice that, in view of the aforementioned analysis, the potential functions $\varphi_i(x)$, which constitute the basic tool for the proposed methodology, are not correct-by-construction navigation functions, despite the potential similarities of the gain tuning for the manipulation of the critical points. Therefore, mainly due to the 2nd order systems that are considered in this work, the tedious procedure of proving that a designed potential field is a navigation function, which was introduced in [22] and is followed in many works, is avoided in this work, simplifying thus the corresponding technical details.

V. SIMULATION RESULTS

To demonstrate the efficiency of the proposed control protocol, we consider a simulation example with $N = 4$, $\mathcal{V} = \{1, 2, 3, 4\}$ spherical agents of the form (1), with $r_i = 0.25\text{m}$ and $d_i = 5\text{m}, \forall i \in \{1, \dots, 4\}$. The initial conditions are set to $p_1(0) = [-3, 0, 5]^\top \text{m}$, $p_2(0) = [-1, 4, 4]^\top \text{m}$, $p_3(0) = [-3, 4, 2]^\top \text{m}$, $p_4(0) = [-4, 3, 6]^\top \text{m}$, $q_1(0) = q_2(0) = q_3(0) = q_4(0) = [0, 0, 0]^\top \text{r}$, which imply the initial neighboring sets $\mathcal{N}_1(0) = \{2\}, \mathcal{N}_2(0) = \{1, 3, 4\}, \mathcal{N}_3(0) = \{2, 4\}, \mathcal{N}_4(0) = \{2, 3\}$. The desired formation is defined by the feasible displacements $p_{12,\text{des}} = -p_{21,\text{des}} = [-1, -1, -2]^\top \text{m}$, $p_{23,\text{des}} = -p_{32,\text{des}} = [-2, -3, 0]^\top \text{m}$, $p_{24,\text{des}} = -p_{42,\text{des}} = [-1, -2, 0]^\top \text{m}$, $p_{34,\text{des}} = -p_{43,\text{des}} = [1, 1, 0]^\top \text{m}$, $q_{12,\text{des}} = -q_{21,\text{des}} = [-\frac{\pi}{4}, 0, -\frac{\pi}{4}]^\top \text{r}$, $q_{23,\text{des}} = -q_{32,\text{des}} = [-\frac{\pi}{12}, 0, 0]^\top \text{m}$, $q_{24,\text{des}} = -q_{42,\text{des}} = [-\frac{\pi}{8}, 0, 0]^\top \text{r}$, $q_{34,\text{des}} = -q_{43,\text{des}} = [\frac{5\pi}{24}, 0, 0]^\top \text{r}$. We consider a workspace of radius $r_w = 10\text{m}$ containing two spherical static obstacles at

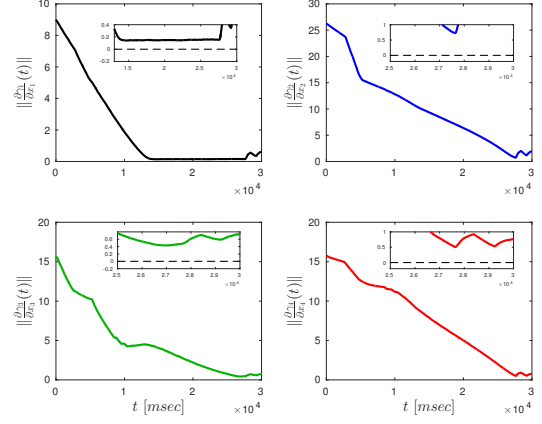


Fig. 7: The evolution of the gradient norms $\|\nabla_{x_i} \gamma_i\|, \forall i \in \{1, \dots, 4\}$, which are shown to be always positive.

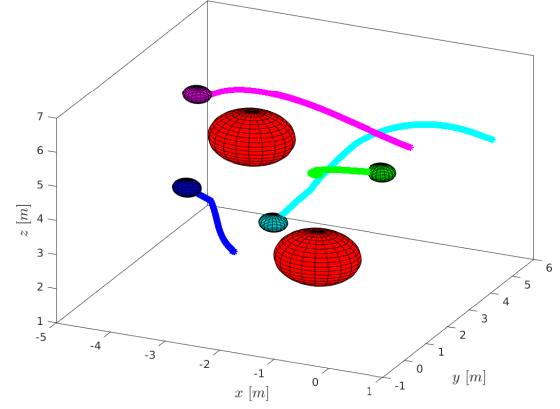


Fig. 8: The motion of the agents in the workspace for $t \in [0, 20]\text{s}$. $p_{o1} = [-3, 3, 5]^\top \text{m}$, $p_{o2} = [-1, 1, 3]^\top \text{m}$ with radii $r_{o1} = r_{o2} = 0.75\text{m}$. An illustration of the workspace with the agents at $t = 0$ is given in Fig. 3. The controller parameters were chosen as $\rho_{ij^*} = 5, k_i = 50, \tilde{k}_i = 50, \tilde{c}_i = 1, \forall i \in \{1, \dots, 4\}$. The simulation results are depicted in Fig. 5-8 for $t \in [0, 30]\text{s}$. In particular, Fig. 5 shows the evolution of the goal functions $\gamma_i, \forall i \in \{1, \dots, 4\}$, which are decreasing to zero, whereas Fig. 6 and 7 depict the obstacle functions β_i and the gradient norms $\|\nabla_{x_i} \gamma_i\|, \forall i \in \{1, \dots, 4\}$, which stay always positive. Furthermore, the control inputs are shown in Fig. 4. Finally, the navigation of the agents in the workspace is pictured in Fig. 8. As proven in the theoretical analysis, the formation is successfully achieved and all the specifications of Problem 1 are met.

VI. CONCLUSIONS AND FUTURE WORK

In this work we proposed a potential function-based decentralized control protocol for multi-agent systems which guarantees formation control with inter-agent collision avoidance, collision avoidance between the agents and the obstacles/workspace boundary, connectivity maintenance as well as singularity avoidance of multiple rigid bodies. Simulation results have verified the validity of the proposed approach. Future efforts will be devoted towards developing a tuning-free control protocol as well as real-time experiments.

APPENDIX I

PROOF OF LEMMA 1

Consider a neighbor $j_i^* \in \mathcal{N}_i(0)$ of agent $i \in \mathcal{V}$, with $\rho_{ij} = 1, \forall j \in \mathcal{N}_i(0) \setminus \{j^*\}$. In order for $\nabla_{x_i} \gamma_i$ to be zero in the set $\hat{\Xi}_i$ it must hold that $\rho_{ij_i^*} \|x_i - x_{j_i^*} - x_{ij_i^*, \text{des}}\| = \left\| \sum_{j \in \mathcal{N}_i(0) \setminus j_i^*} (x_i - x_j - x_{ij, \text{des}}) \right\|$, for some $x \in \hat{\Xi}_i$. Therefore, a sufficient condition for the above expression not to hold for every $x \in \hat{\Xi}_i$ is $\rho_{ij_i^*} \|x_i - x_{j_i^*} - x_{ij_i^*, \text{des}}\| > \left\| \sum_{j \in \mathcal{N}_i(0) \setminus j_i^*} (x_i - x_j - x_{ij, \text{des}}) \right\|$. Since $\|x_i - x_{j_i^*} - x_{ij_i^*, \text{des}}\| = \sqrt{\gamma_{ij_i^*}} \geq \sqrt{\mu_{ij_i^*}}$ in the set $\hat{\Xi}_i$, we design $\rho_{ij_i^*}$ such that $\rho_{ij_i^*} \geq \bar{\rho}_i$, where $\bar{\rho}_i$ is given in (8), and we provide hence a lower bound of $\|\nabla_{x_i} \gamma_i(x)\|$ when $x \in \hat{\Xi}_i$, as:

$$\begin{aligned} & \|\nabla_{x_i} \gamma_i(x)\| \\ &= \|\rho_{ij_i^*} (x_i - x_{j_i^*} - x_{ij_i^*, \text{des}}) + \sum_{j \in \mathcal{N}_i(0)} (x_i - x_j - x_{ij, \text{des}})\| \\ &\geq \bar{\rho}_i \|x_i - x_{j_i^*} - x_{ij_i^*, \text{des}}\| - \left\| \sum_{j \in \mathcal{N}_i(0)} (x_i - x_j - x_{ij, \text{des}}) \right\| \geq \tilde{\rho}_i > 0, \forall x \in \hat{\Xi}_i. \end{aligned}$$

APPENDIX II

PROOF OF LEMMA 2

By introducing the notation $x_i^* = [p_i^\top, 0, 0, 0]^\top \in W, i \in \mathcal{V}$, we have that:

$$\begin{aligned} \nabla_{x_i} (\eta_{ij,a}) &= \nabla_{x_i} (\|p_i - p_j\|^2) = 2(x_i^* - x_j^*), \\ \nabla_{x_i} (\eta_{iz,o}) &= \nabla_{x_i} (\|p_i - p_{o_z}\|^2) = 2x_i^*, \\ \nabla_{x_i} (\eta_{il,c}) &= \nabla_{x_i} (-\|p_i - p_l\|^2) = -2(x_i^* - x_l^*), \\ \frac{\partial b_{ij,a}}{\partial \eta_{ij,a}} &= \begin{cases} \frac{\phi_{i,a}}{\partial \eta_{ij,a}}, & 0 \leq \eta_{ij,a} < d_i^2 - \underline{d}_{ij,a}^2, \\ 0, & d_i^2 - \underline{d}_{ij,a}^2 \leq \eta_{ij,a}, \end{cases}, \\ \frac{\partial b_{iz,o}}{\partial \eta_{iz,o}} &= \begin{cases} \frac{\phi_{i,o}}{\partial \eta_{iz,o}}, & 0 \leq \eta_{iz,o} < d_i^2 - \underline{d}_{iz,o}^2, \\ 0, & d_i^2 - \underline{d}_{iz,o}^2 \leq \eta_{iz,o}, \end{cases}, \\ \frac{\partial b_{ij,c}}{\partial \eta_{ij,c}} &= \begin{cases} 0, & \eta_{ij,c} < 0, \\ \frac{\phi_{i,c}}{\partial \eta_{ij,c}}, & 0 \leq \eta_{ij,c} < d_i^2 - \underline{d}_{ij,a}^2, \\ 0, & d_i^2 - \underline{d}_{ij,a}^2 \leq \eta_{ij,c}, \end{cases}, \\ \nabla_{x_i} (b_{ij,a}) &= 2 \left(\frac{\partial b_{ij,a}}{\partial \eta_{ij,a}} \right) (x_i^* - x_j^*), \\ \nabla_{x_i} (b_{iz,o}) &= 2 \left(\frac{\partial b_{iz,o}}{\partial \eta_{iz,o}} \right) x_i^*, \\ \nabla_{x_i} (b_{ij,c}) &= -2 \left(\frac{\partial b_{ij,c}}{\partial \eta_{ij,c}} \right) (x_i^* - x_j^*), \\ \nabla_{x_i} (b_{iw}) &= -4 \left(1 - \frac{\|p_i\|^2}{(r_w - r_i)^2} \right) x_i^*, \\ \nabla_{x_i} (b_{J_i}) &= [0, 0, 0, 0, -2 \sin(\theta_i), 0]^\top. \end{aligned}$$

Since $\beta_i(x) \geq \beta > 0, \forall x \in \hat{\Xi}_i \subseteq W^N$, we conclude from (5d) that $\|p_i\|^2 < (r_w - r_i)^2$ and hence $b_{iw} \leq 1, \forall i \in \mathcal{V}$ and also all the aforementioned derivatives are bounded. Therefore, in view of (5), we conclude that $\beta_i(x) \leq 1, \forall i \in \mathcal{V}$, which implies that $b_{ij,a}, b_{iz,o}, b_{ij,c}, b_{iw}, b_{J_i} \geq \beta > 0$. Hence, the term $\nabla_{x_i} [\beta_i(x)]$, which is given by a sum of products of the aforementioned derivatives with bounded terms of β_i , is bounded by a positive constant $\bar{\beta}_i$.

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